A recipe for enriched categories

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What is an enriched category?

- A collection of objects $|\mathcal{C}|$.
- For each pair of objects x, y, an object $\mathcal{C}(x,y)$ in some category $\mathcal{V}$.
- Identities and composites, satisfying associativity and unitality laws.

$$I \xrightarrow{1_x} \mathcal{C}(x,x) \qquad \mathcal{C}(y,z), \mathcal{C}(x,y) \xrightarrow{\circ_{x,y,z}} \mathcal{C}(x,z)$$

For each monoidal category $\mathcal{V}$, we have the notion of:

- $\mathcal{V}$-enriched category
- $\mathcal{V}$-enriched functor
- $\mathcal{V}$-enriched distributor
- $\mathcal{V}$-enriched natural transformation

Together, these assemble into a structure known as a virtual double category.

Virtual double categories

A virtual double category is a structure comprising

- Objects.
- Tight-cells $X \longrightarrow Y$.
- Loose-cells $X' \xrightarrow{\;\mid\;} X$.
- 2-cells

$$\begin{tikzcd}
X_1 \arrow[d, "x"'] & X_2 \arrow[l, "p_1"', "\mid" marking] & \cdots \arrow[l, "p_2"', "\mid" marking] & X_{n-1} \arrow[l, "p_{n-1}"', "\mid" marking] & X_n \arrow[l, "p_n"', "\mid" marking] \arrow[d, "x'"] \\
X & & \chi & & X' \arrow[llll, "p", "\mid" marking]
\end{tikzcd}$$

Virtual double categories

A virtual double category is a structure comprising

these form a category

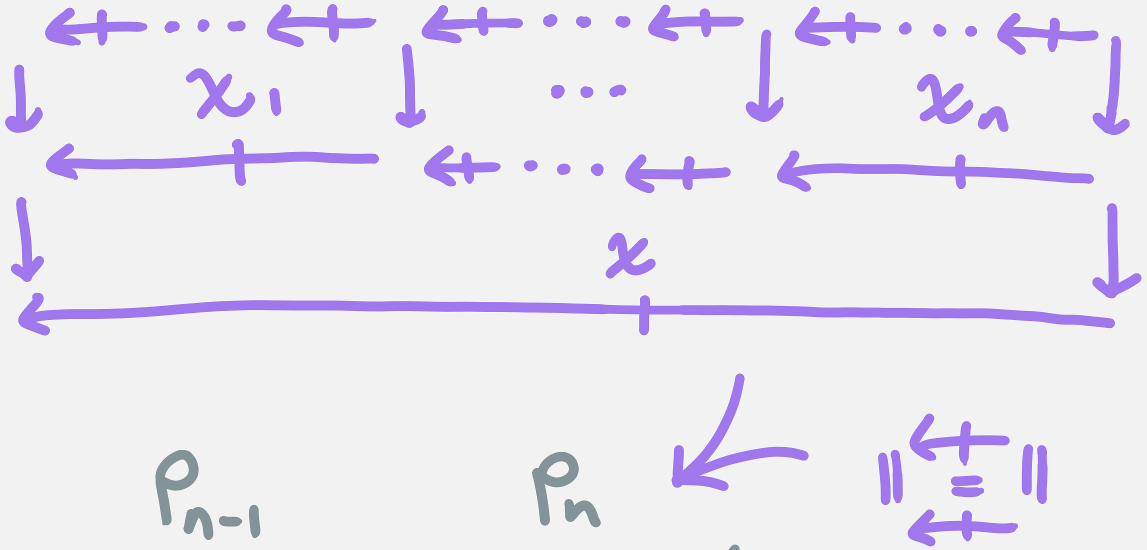
- Objects.
- Tight-cells $X \to Y$.
- Loose-cells $X' \nrightarrow X$.
- 2-cells

$$
\begin{tikzcd}
X_1 \arrow[d, "x"'] & X_2 \arrow[l, "p_1"', "|" marking] & \cdots \arrow[l, "p_2"', "|" marking] & X_{n-1} \arrow[l, "p_{n-1}"', "|" marking] & X_n \arrow[l, "p_n"', "|" marking] \arrow[d, "x'"] \\
X \arrow[rrrr, phantom, "\chi", shift left=4ex] & & & & X' \arrow[llll, "p", "|" marking]
\end{tikzcd}
$$

Virtual double categories

A virtual double category is a structure comprising

- Objects.
- Tight-cells $X \longrightarrow Y$.
- Loose-cells $X' \overset{\shortmid}{\longrightarrow} X$.
- 2-cells



$$\begin{tikzcd}
X_1 \arrow[d, "x"'] & X_2 \arrow[l, "\shortmid"{marking}, "p_1"'] & \cdots \arrow[l, "\shortmid"{marking}, "p_2"'] & X_{n-1} \arrow[l, "\shortmid"{marking}, "p_{n-1}"'] & X_n \arrow[l, "\shortmid"{marking}, "p_n"'] \arrow[d, "x'"] \\
X \arrow[rr, phantom, "\chi"] & & & & X' \arrow[llll, "\shortmid"{marking}, "p"]
\end{tikzcd}$$

Normal VDCs

A VDC is normal if it admits loose-identities, i.e. for each object X, there is a loose-cell

$$\begin{tikzcd} X \arrow[r, "X(1,1)", "\shortmid"{marking}] & X \end{tikzcd}$$

such that $(n+1)$-ary 2-cells from

$$\begin{tikzcd} \cdot \arrow[r, "\shortmid"{marking}] & \cdots \arrow[r, "\shortmid"{marking}] & X \arrow[r, "X(1,1)", "\shortmid"{marking}] & X \arrow[r, "\shortmid"{marking}] & \cdots \arrow[r, "\shortmid"{marking}] & \cdot \end{tikzcd}$$

are in natural bijection with n-ary 2-cells from

$$\begin{tikzcd} \cdot \arrow[r, "\shortmid"{marking}] & \cdots \arrow[r, "\shortmid"{marking}] & X \arrow[r, "\shortmid"{marking}] & \cdots \arrow[r, "\shortmid"{marking}] & \cdot \end{tikzcd}$$

$\mathcal{V}$-distributors & $\mathcal{V}$-natural transformations

- A $\mathcal{V}$-distributor $\mathcal{X} \overset{\rho}{\longrightarrow} \mathcal{Y}$ comprises an object $\rho(y,x) \in \mathcal{V}$ $(x \in |\mathcal{X}|,\ y \in |\mathcal{Y}|)$, together with pre- and postcomposition operations.

$$\mathcal{Y}(y',y), \rho(y,x) \to \rho(y',x) \qquad \rho(y,x), \mathcal{X}(x,x') \to \rho(y,x')$$

$\mathcal{V}$-distributors & $\mathcal{V}$-natural transformations

- A $\mathcal{V}$-distributor $\mathbb{X} \xrightarrow{p} \mathbb{Y}$ comprises an object $p(y,x) \in \mathcal{V}$ $(x \in |\mathbb{X}|,\ y \in |\mathbb{Y}|)$, together with pre- and postcomposition operations.

$$\mathbb{Y}(y',y), p(y,x) \to p(y',x) \qquad p(y,x), \mathbb{X}(x,x') \to p(y,x')$$

- A $\mathcal{V}$-natural transformation

$$\begin{tikzcd} \mathbb{X}_1 \arrow[d, "f"'] \arrow[drrr, phantom, "\varphi"] & \cdots \arrow[l, "p_1"'] & & \mathbb{X}_n \arrow[ll, "p_n"'] \arrow[d, "g"] \\ \mathbb{Y} & & & \mathbb{Y}' \arrow[lll, "q"] \end{tikzcd}$$

comprises a family

$$p_1(x_1,x_2), \ldots, p_n(x_{n-1},x_n) \to q(fx_1, gx_n)$$

The construction of a VDC of enriched categories from a monoidal category is functorial, and defines a 2-functor:

$$(-)\text{-}\mathbb{C}\mathrm{at} : \mathrm{MonCat} \longrightarrow \mathrm{VDblCat}$$

from the 2-category of monoidal categories, lax monoidal functors, and monoidal natural transformations to the 2-category of virtual double categories, functors, and transformations.

However, this definition comes to us prepackaged. How could we arrive at this definition ourselves?

For instance, is there any sense in which this construction is canonical?

In other words, what evidence do we have that we have a good definition of enriched category?

Categories as monads

A small category can be defined as a monad in the bicategory of spans,

$$\mathcal{C}_0 \xleftarrow{s} \mathcal{C}_1 \xrightarrow{t} \mathcal{C}_0$$

or as a monad in the bicategory of matrices.

$$\mathcal{C}_0 \times \mathcal{C}_0 \longrightarrow \mathrm{Set}$$

Categories as monads

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or as a monad in the bicategory of matrices.

$$\mathcal{C}_0 \times \mathcal{C}_0 \longrightarrow \mathrm{Set}$$

Similarly, small $\mathcal{V}$-categories can be defined as monads in the bicategory of $\mathcal{V}$-matrices.

$$\mathcal{C}_0 \times \mathcal{C}_0 \longrightarrow \mathcal{V}$$

Therefore, to understand the extent to which the enriched category construction is canonical, it suffices to understand:

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2. The extent to which the monad construction is canonical.

Therefore, to understand the extent to which the enriched category construction is canonical, it suffices to understand:

1. The extent to which the enriched matrix construction is canonical.
2. The extent to which the monad construction is canonical.
3. How these two constructions interact.

A one-dimensional intuition

The following analogue will be helpful to keep in mind.

- A category is cocomplete iff it admits coproducts and reflexive coequalisers.

A one-dimensional intuition

The following analogue will be helpful to keep in mind.

- A category is cocomplete iff it admits coproducts and reflexive coequalisers.

- The free cocompletion of a category is given by first freely adding coproducts and then coequalisers of pseudo-equivalence relations.

We shall study the canonicity of the $(-)\text{-}\mathbb{C}\mathrm{at}$ construction.

While it is possible to talk about universality for operations that do not have the same codomain as their domain, it is easier for operations that do.

We shall enhance our definition of enriched category accordingly.

Categories enriched in a virtual double category

Let $\mathbb{V}$ be a VDC. A $\mathbb{V}$-category comprises:

- A collection of objects $|\mathcal{C}|$.
- For each object x, an extent $\underline{x} \in \mathbb{V}$.
- For each pair of objects x, y, a loose-cell $y \xrightarrow{\mathcal{C}(x,y)} \underline{x}$ in $\mathbb{V}$.

- For each object x, a 2-cell in $\mathbb{V}$.

$$
\begin{tikzcd}
\underline{x} \arrow[r, equal] \arrow[d, equal] \arrow[dr, phantom, "\iota_x"] & \underline{x} \arrow[d, equal] \\
\underline{x} \arrow[r, "|"{marking}, "\mathcal{C}(x,x)"'] & \underline{x}
\end{tikzcd}
$$

- For each triple x, y, z, a 2-cell in $\mathbb{V}$.

$$
\begin{tikzcd}
\underline{z} \arrow[r, "|"{marking}, "\mathcal{C}(y,z)"] \arrow[d, equal] \arrow[drr, phantom, "\circ_{x,y,z}"] & y \arrow[r, "|"{marking}, "\mathcal{C}(x,y)"] & \underline{x} \arrow[d, equal] \\
\underline{z} \arrow[rr, "|"{marking}, "\mathcal{C}(x,z)"'] & & \underline{x}
\end{tikzcd}
$$

Satisfying associativity and unitality axioms.

The construction of a VDC of enriched categories from a VDC is functorial, and defines a 2-functor:

$$(-)\text{-}\mathbb{C}\mathbf{at} : \mathbf{VDblCat} \longrightarrow \mathbf{VDblCat}$$

from the 2-category of virtual double categories, functors, and transformations to itself.

The universality of the $(-)$-$\mathbb{C}\mathrm{at}$ construction has been studied previously, in a special case, by Garner & Shulman.

We will take inspiration from their characterisation.

Garner & Shulman showed that, for $\mathbb{V}$ a locally cocomplete pseudo double category with companions,

$\mathbb{V}$-$\mathbb{C}$at is equivalently:

- The free cocompletion of $\mathbb{V}$ under collages.
- The free cocompletion of $\mathbb{V}$ under coproducts & collapses.
- The free cocompletion of $\mathbb{V}$ under lax colimits of lax functors.

I will explain how we can drop essentially all of their assumptions.

In particular, this permits us to exhibit a universal property of $\mathcal{V}$-$\mathbf{Cat}$ when $\mathcal{V}$ is an arbitrary monoidal category, with no cocompleteness or closure assumptions.

Roadmap

1. The $(-)$-$\mathbb{S}\mathrm{et}$ construction.
2. The $\mathbb{M}\mathrm{nd}$ construction.
3. The $(-)$-$\mathbb{C}\mathrm{at}$ construction.
4. Local colimits.

Coproducts in a VDC

A VDC $\mathbb{V}$ admits coproducts if:

1. Its underlying category admits coproducts.
2. For each family of loose-cells in $\mathbb{X}$

$$\left\{ X(x) \xrightarrow{p(y,x)} Y(y) \right\}_{\substack{x \in |X|, \\ y \in |Y|}}$$

$$\begin{tikzcd} {|X|} \arrow[dr, "x"'] & & {|Y|} \arrow[dl, "y"] \\ & {|\mathbb{V}|} & \end{tikzcd}$$

there is a loose-cell

$$\coprod X \xrightarrow{\coprod p} \coprod Y$$

and 2-cells

$$\begin{tikzcd} X(x) \arrow[r, "{p(y,x)}", "\shortmid"{marking}] \arrow[d, "\amalg_x"'] \arrow[dr, phantom, "\amalg_{x,y}"] & Y(y) \arrow[d, "\amalg_y"] \\ \coprod X \arrow[r, "\coprod p"', "\shortmid"{marking}] & \coprod Y \end{tikzcd}$$

such that, for each family of 2-cells

$$\begin{tikzcd} X_1(x_1) \arrow[r, "{p_1(x_2,x_1)}", "\shortmid"{marking}] \arrow[d, "f_{x_1}"'] & \cdots \arrow[r, "{p_n(x_n,x_{n-1})}", "\shortmid"{marking}] \arrow[d, phantom, "\varpi_{\vec{x}}"] & X_n(x_n) \arrow[d, "g_{x_n}"] \\ A \arrow[rr, "q"', "\shortmid"{marking}] & & B \end{tikzcd}$$

there exists a unique 2-cell

$$\begin{tikzcd} \coprod X_1 \arrow[r, "\coprod p_1", "|"'] \arrow[d, "{[f_x]_x}"'] & \cdots \arrow[r, "\coprod p_n", "|"'] \arrow[d, phantom, "{[\omega_{\vec{x}}]_{\vec{x}}}"] & \coprod X_n \arrow[d, "{[g_x]_x}"] \\ A \arrow[rr, "|", "q"'] & & B \end{tikzcd}$$

factoring each $\overline{\omega}_{\vec{x}}$.

The enriched set construction

Let $\mathbb{V}$ be a VDC. We define a VDC $\mathbb{V}\text{-}\mathbb{S}\mathrm{et}$:

- Objects are families of objects of $\mathbb{V}$.
- A tight-cell from X to Y comprises a function

$$|X| \xrightarrow{f} |Y|$$

  and, for each $x \in |X|$, a tight-cell

$$f_x : X(x) \longrightarrow Y(f(x))$$

- A loose-cell from X to Y comprises

$$\{ p(y,x) : X(x) \nrightarrow Y(y) \}_{x \in |X|,\, y \in |Y|}$$

- A 2-cell

$$
\begin{tikzcd}
X_1 \arrow[r, "p_1"] \arrow[d, "f"'] & \cdots \arrow[r, "p_n"] \arrow[d, phantom, "\varphi"] & X_n \arrow[d, "g"] \\
Y \arrow[rr, "q"'] & & Y'
\end{tikzcd}
$$

is a family of 2-cells in $\mathbb{V}$

$$
\begin{tikzcd}
X_1(x_1) \arrow[r, "p_1(x_2,x_1)"] \arrow[d, "f_{x_1}"'] & \cdots \arrow[r, "p_n(x_n,x_{n-1})"] \arrow[d, phantom, "\varphi_{\vec{x}}"] & X_n(x_n) \arrow[d, "g_{x_n}"] \\
Y(f(x_1)) \arrow[rr, "q(g(x_n),f(x_1))"'] & & Y'(g(x_n))
\end{tikzcd}
$$

Lemma

$(-)$-$\mathbb{S}\mathrm{et}$ is a lax-idempotent pseudomonad on $\mathbb{V}\mathrm{DblCat}$.

Lemma

$(-)$-$\mathbb{S}\mathrm{et}$ is a lax-idempotent pseudomonad on $\mathbf{VDblCat}$.

Theorem Let $\mathbb{V}$ be a VDC.

$\mathbb{V}$-$\mathbb{S}\mathrm{et}$ is the free cocompletion of $\mathbb{V}$ under coproducts.

The loose-monads construction

Let $\mathbb{X}$ be a VDC. We define a VDC $\mathbb{M}\mathrm{nd}(\mathbb{X})$:

- Objects are loose-monads.
- Tight-cells are monad morphisms.
- Loose-cells are monad bimodules.
- 2-cells are monad transformations.

$$
\begin{tikzcd}
X \arrow[r, equal] \arrow[d, equal] \arrow[dr, phantom, "\eta"] & X \arrow[d, equal] \\
X \arrow[r, "T"', "\shortmid" marking] & X
\end{tikzcd}
\qquad
\begin{tikzcd}
X \arrow[r, "T", "\shortmid" marking] \arrow[d, equal] \arrow[drr, phantom, "\mu"] & X \arrow[r, "T", "\shortmid" marking] & X \arrow[d, equal] \\
X \arrow[rr, "T"', "\shortmid" marking] & & X
\end{tikzcd}
$$

Colimits of loose-monads

In a double category, we can take the colimit of a loose-cell $p : X \nrightarrow Y$, which is given by an object $\overline{p}$, tight-cells $\amalg_X : X \to \overline{p}$ and $\amalg_Y : Y \to \overline{p}$, and a 2-cell

$$\begin{tikzcd} X \arrow[r, "p", "|" marking] \arrow[d, "\amalg_X"'] \arrow[dr, phantom, "\amalg_p"] & Y \arrow[d, "\amalg_Y"] \\ \overline{p} \arrow[r, equal, "|" marking] & \overline{p} \end{tikzcd}$$

universal in a suitable sense. This is called the cotabulator of p.

When the loose-cell in question has additional structure, e.g. of a loose-monad, it is natural to ask that the colimit respects this structure.

This leads to the notion of a collapse of a loose-monad, which is another kind of double categorical colimit.

Collapses in a VDC

Let $X \xrightarrow{T} X$ be a loose-monad. A **collapse** of T is an object $⟪T⟫$ together with a tight-cell and 2-cell

$$\begin{tikzcd} X \arrow[r, "T", "\shortmid"{marking}] \arrow[d, "\amalg_X"'] \arrow[dr, phantom, "\amalg_T"] & X \arrow[d, "\amalg_X"] \\ ⟪T⟫ \arrow[r, equal, "\shortmid"{marking}] & ⟪T⟫ \end{tikzcd}$$

which is universal amongst loose-monad morphisms.

Let $S \overset{m}{\nrightarrow} T$ be a loose-monad bimodule.

A collapse of m is a loose-cell

$$\langle\!\langle S \rangle\!\rangle \xrightarrow[\;\;+\;\;]{\langle\!\langle m \rangle\!\rangle} \langle\!\langle T \rangle\!\rangle$$

together with a 2-cell

$$\begin{tikzcd}
X \arrow[r, "m", "\shortmid"{marking}] \arrow[d, "\amalg_X"'] \arrow[dr, phantom, "\amalg_m"] & Y \arrow[d, "\amalg_Y"] \\
\langle\!\langle S \rangle\!\rangle \arrow[r, "\langle\!\langle m \rangle\!\rangle"', "\shortmid"{marking}] & \langle\!\langle T \rangle\!\rangle
\end{tikzcd}$$

which is universal amongst loose-monad transformations.

$\mathbb{M}\mathrm{nd}$ is a normal morphism coclassifier

There is a 2-adjunction [CS10]:

$$\begin{tikzcd} \mathbf{VDblCat}_n \arrow[d, ""{name=A}] \\ \mathbf{VDblCat} \arrow[u, bend right, dashed, "\mathbb{M}\mathrm{nd}"', ""{name=B}] \arrow[from=A, to=B, phantom, "\dashv"] \end{tikzcd}$$

Furthermore, this 2-adjunction is lax-idempotent. Consequently, $\mathbb{M}\mathrm{nd}$ is a lax-idempotent 2-monad on $\mathbf{VDblCat}_n$.

Theorem Let $\mathbb{X}$ be a normal VDC. $\mathbb{M}\mathrm{nd}(\mathbb{X})$ is the free cocompletion of $\mathbb{X}$ under collapses.

The enriched category construction

We define a lax-idempotent pseudomonad by composition:

$$(-)\text{-}\mathbb{C}\mathrm{at} := \begin{tikzcd} (-)\text{-}\mathbb{S}\mathrm{et} \\ \mathrm{VDblCat} \arrow[loop above] \arrow[d, shift left, "\mathbb{M}\mathrm{nd}"] \arrow[d, phantom, "\dashv"] \\ \mathrm{VDblCat}_n \arrow[u, shift left] \end{tikzcd}$$

Theorem Let $\mathbb{V}$ be a normal VDC. $\mathbb{V}$-$\mathbb{C}\mathrm{at}$ is the free cocompletion of $\mathbb{V}$ under collages.

Theorem Let $\mathbb{V}$ be a normal VDC. $\mathbb{V}$-$\mathbb{C}\mathrm{at}$ is the free cocompletion of $\mathbb{V}$ under collages.

For a VDC $\mathbb{V}$, a $\mathbb{V}$-category can be viewed as a diagram in $\mathbb{V}$, i.e. a functor into $\mathbb{V}$. A collage is a colimit of such a diagram.

Placing coproducts & collapses on an equal footing

Given that $(-)\text{-}\mathbb{S}\mathrm{et}$ freely adds coproducts and $\mathbb{M}\mathrm{nd}$ freely adds collapses, we might expect that $(-)\text{-}\mathbb{C}\mathrm{at}$ freely adds coproducts and collapses.

However, there is a subtlety. $\mathbb{M}\mathrm{nd}$ is a free construction on normal VDCs, whereas $(-)\text{-}\mathbb{S}\mathrm{et}$ only produces a VDC. In particular, the inclusion $\mathbb{V} \longrightarrow \mathbb{V}\text{-}\mathbb{S}\mathrm{et}$ is not always normal.

Local colimits

Let $\mathbb{X}$ be a VDC. A **local colimit** of a functor $J \xrightarrow{D} \mathbb{X}[\![X, Y]\!]$ comprises a loose-cell

$$X \xrightarrow[\mid]{\operatorname{colim} D} Y$$

together with 2-cells

$$\begin{tikzcd} X \arrow[r, "D_j", "\mid"{marking}] \arrow[d, equal] \arrow[dr, phantom, "\amalg_j"] & Y \arrow[d, equal] \\ X \arrow[r, "\operatorname{colim} D"', "\mid"{marking}] & Y \end{tikzcd}$$

such that, for every family of 2-cells

$$\begin{tikzcd}
\cdot \arrow[r, "|" marking] \arrow[d] & \cdots \arrow[r, "|" marking] & X \arrow[r, "D_j", "|" marking] \arrow[dr, phantom, "\varpi_j"] & Y \arrow[r, "|" marking] & \cdots \arrow[r, "|" marking] & \cdot \arrow[d] \\
A \arrow[rrrrr, "p"', "|" marking] & & & & & B
\end{tikzcd}$$

there exists a unique 2-cell

$$\begin{tikzcd}
\cdot \arrow[r, "|" marking] \arrow[d] & \cdots \arrow[r, "|" marking] & X \arrow[r, "\operatorname{colim} D", "|" marking] \arrow[dr, phantom, "{[\varpi_j]_j}"] & Y \arrow[r, "|" marking] & \cdots \arrow[r, "|" marking] & \cdot \arrow[d] \\
A \arrow[rrrrr, "p"', "|" marking] & & & & & B
\end{tikzcd}$$

factoring each ϖ_j.

Lemma

Let $\mathbb{V}$ be a normal VDC. If $\mathbb{V}$ admits local initial objects, then $\mathbb{V}$-$\mathbb{S}\mathrm{et}$ is normal.

Furthermore, $\mathbb{V}$-$\mathbb{S}\mathrm{et}$ admits those local colimits that $\mathbb{V}$ does.

Lemma

Let $\mathbb{V}$ be a normal VDC. If $\mathbb{V}$ admits local initial objects, then $\mathbb{V}\text{-}\mathbb{S}\mathrm{et}$ is normal.

Furthermore, $\mathbb{V}\text{-}\mathbb{S}\mathrm{et}$ admits those local colimits that $\mathbb{V}$ does.

Consequently, $(-)\text{-}\mathbb{S}\mathrm{et}$ lifts to a pseudomonad on $\mathbf{VDblCat}_{n,o}$.

Lemma

Let $\mathbb{X}$ be a normal VDC. $\mathbb{M}\mathrm{nd}(\mathbb{X})$ admits those local colimits that $\mathbb{X}$ does.

Consequently, $\mathbb{M}\mathrm{nd}$ lifts to a 2-monad on $\mathbf{VDblCat}_{n,o}$.

Theorem

$(-)\text{-}\mathbb{S}\mathrm{et}$ pseudodistributes over $\mathbb{M}\mathrm{nd}$, and the composite is $(-)\text{-}\mathbb{C}\mathrm{at}$.

$$\begin{tikzcd}
\mathrm{VDblCat}_{n,o} \arrow[r, "(-)\text{-}\mathbb{S}\mathrm{et}"] \arrow[rr, bend right, "(-)\text{-}\mathbb{C}\mathrm{at}"'] & \mathrm{VDblCat}_{n,o} \arrow[r, "\mathbb{M}\mathrm{nd}"] & \mathrm{VDblCat}_{n,o}
\end{tikzcd}$$

Theorem Let $\mathbb{V}$ be a normal VDC with local initial objects. $\mathbb{V}$-$\mathbb{C}\mathrm{at}$ is the free cocompletion of $\mathbb{V}$ under coproducts & collapses.

Representability

We wish to recover the characterisation of [GS16], for which one ingredient is missing.

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We wish to recover the characterisation of [GS16], for which one ingredient is missing.

A VDC is representable if it is normal and admits binary loose-composites, equivalently if it is a pseudo double category.

Lemma Let $\mathbb{V}$ be a PDC with local coproducts. Then so is $\mathbb{V}$-$\mathbb{S}\mathrm{et}$.

Lemma Let $\mathbb{X}$ be a PDC with local reflexive coequalisers. Then so is $\mathbb{M}\mathrm{nd}(\mathbb{X})$.

Corollary The pseudodistributive law between $(-)$-$\mathbb{S}\mathrm{et}$ and $\mathbb{M}\mathrm{nd}$ lifts to locally cocomplete pseudo double categories.

A recipe for enriched categories

1. Take your favourite monoidal category $\mathcal{V}$, viewed as a VDC with one object.
2. If $\mathcal{V}$ has an initial object preserved by $\otimes$ on both sides, freely add coproducts & collapses.

   Otherwise, freely add collages.
3. Let the mixture rest for 15 minutes.
4. Enjoy your delicious $\mathcal{V}$-$\mathbf{Cat}$!

Postscript: $\mathcal{V}$-normed categories

A $\mathcal{V}$-normed category is a category enriched in $\mathrm{Fam}(\mathcal{V})$, the category of families of $\mathcal{V}$.

In [P24], Patterson introduced a double categorical $\mathbb{F}\mathrm{am}$ construction.

Theorem $\mathbb{F}\mathrm{am} \simeq \mathbb{LC}(-)\text{-}\mathbb{S}\mathrm{et}$, where $\mathbb{LC}$ is the local coproduct completion.

Corollary Loose-monads in $\mathbb{F}\mathrm{am}(\mathbb{V})$ are precisely $\mathbb{V}$-normed categories.